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THE HEAD-TAIL EFFECT IN NAL BOOSTER

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INTRODUCTION

In this paper we investigate the possibility that the "missing bunches" phenomena[1,2] observed in the booster might be caused by the space-charge effect called "head-tail" effect. This has been discovered in Adone,[3] the 1.5 GeV electron-positron storage ring in Frascati, and theoretically analyzed by C. Pellegrini.[4] A more understandable and plain mathematical approach to describe the phenomena can be found in M. Sand's paper.[5] In our work we shall essentially apply the results in the form as presented in the last paper.

We had to extrapolate the theory to the case where the energy of the beam, and all the quantities related to this, change with time. Moreover, we had to work out an analytic expression describing the coupling between a particle executing transverse oscillations and the lamination of the magnets, which, in our case, is the closest item to the beam. This is done in Section 2 of this paper, whereas Section 1 is a description of the "head-tail" effect by means of the two-particles model, which avoids involvement in too many mathematical details.

Section 3 is devoted to the calculation of the behavior of the booster beam during the acceleration cycle. In this calculation the adiabatic damping of both transverse and longitudinal

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oscillations of a particle is neglected, namely, the size and the length of a beam bunch are supposed constant during the acceleration cycle.

Finally, in Section 4 we put down some numbers. We found a good agreement between theory and experiments, assuming that the instability of the beam is at the mode $n = 0$, which corresponds to oscillations of the center of mass of the beam. Indeed, coherent oscillations few milliseconds before the transition energy have been observed in the booster.[6]

1. THE TWO PARTICLE-MODEL*

Let us consider two particles in the same beam bunch. They execute synchrotron oscillations with the same amplitude but opposite phases. One half of the time, one particle is at the "head" of the bunch and the other particle at the "tail". The other half of the time, the relative position is exchanged. We describe the synchrotron oscillations in terms of the time displacement τ, i.e., the distance in unit of time from the synchronous particle with energy E_s, and the energy deviation ΔE from E_s. The relationship between ΔE and τ is

$$\dot{\tau} = -\alpha\frac{\Delta E}{E_s} \tag{1}$$

where α is the "momentum compaction", and is defined later in the paper.

We assume the two particles have zero time deviation at the time $t = 0$, which we may take, for instance, at the injection.

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*Some of this section is inspired by M. Sand's paper.

Now let us introduce the betatron oscillations, say, in the horizontal plane x. Let us explicit the dependence of the betatron (angular) frequency* ω on the energy deviation:

$$\omega = \omega_o + \omega_o(\xi-\alpha)\ \frac{\Delta E}{E_s} = \nu\Omega \tag{2}$$

$\xi = \frac{\Delta\nu/\nu}{\Delta E/E}$, is the "chromaticity" of the machine $\alpha = -\frac{\Delta\Omega/\Omega}{\Delta E/E}$, is the "momentum compaction".

We suppose ω_o is constant and the same for both particles. We suppose, also, the two betatron oscillations have the same amplitude Z and the same phase at the initial time t = 0 when the particles have zero time displacement. The phase difference between the two oscillations at a later time is, then,

$$\Delta\phi = \int_0^t \omega_2(t)dt - \int_0^t \omega_1(t)dt = 2\omega_o \int_0^t \frac{\xi-\alpha}{\alpha}\,\dot{\tau}\ dt \tag{3}$$

where the subscript 1 refers to particle "1" and the subscript 2 refers to particle "2", and

$$\dot{\tau} = \dot{\tau}_1 = -\dot{\tau}_2.$$

Here we are taking ξ and α to be general functions of time.

In the period of time the particle "2" is leading, it leaves a wake field which produces a force on the trailing particle "1" proportional to the displacement of the leading particle at the early time $t - \tau_2 + \tau_1$. This force has the following form

$$F_{21}(t) = x_2(t - \tau_2 + \tau_1)\ \rho\ (\tau_2 - \tau_1) \tag{4}$$

where ρ is a function of the distance between the two particles

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*By frequency we shall always mean the angular frequency.

and describes the decay of the wake field caused by the electro-magnetic properties of the surrounding media.

Since the particle "1" which is oscillating with the same amplitude Z and frequency ω_1, is driven by a synchronous oscillating force with the amplitude $Z\rho$ and the phase lag $\Delta\psi$, the amplitude Z increases at the relative rate

$$\mu(t) = \frac{1}{Z}\frac{dZ}{dt} = \frac{\omega_1 \rho}{2k}\Delta\psi \tag{5}$$

where k is the restoring force of the oscillator, and

$$\begin{aligned}\Delta\psi &= \int^{t-\tau_2+\tau_1} \omega_2 dt - \int^{t} \omega_1 dt = \\ &= \Delta\phi + \int^{t-\tau_2+\tau_1} \omega_2 dt \\ &\approx 2\omega_o \int_o^t \frac{\xi}{\alpha}\dot{\tau}_1\, dt \qquad (\text{mod. } 2\pi)\end{aligned} \tag{6}$$

Of course, Eq. (5) applies only when the particle "1" is trailing. There will be no growth for this particle when it is leading, in which case the amplitude of particle "2" will increase with a growth rate similar to Eq. (5).

From Eqs. (5,6), we see that the beam is stable if the chromaticity ξ is identically zero. In general, we say the motion is stable if

$$\rho\xi/\alpha \geq 0.$$

Vice-versa, the beam is unstable if $\rho\xi/\alpha$ is negative.

In the case we had chosen our initial conditions at t = 0 for the two particles with opposite betatron displacements, the only effect would be to change the sign of our expression for μ.

Henceforth, in this case, the motion of particle "1" would be stable if $\rho\xi/\alpha \leq 0$. Generalizing, the coupling between two particles is described as a linear combination of one "symmetric" and one "asymmetric" mode, which have opposite signs for μ. Thus it might very well be the case where the motion is almost stable at the injection (small $\Delta\psi$) and becomes unstable close to the transition (where presumably $\Delta\psi$ is largest), since the distribution of "symmetric" and "asymmetric" modes is memorized through all the cycle by means of the phase difference function $\Delta\psi$.

Finally, we observe that the quantity of $\rho\xi/\alpha$ changes sign at the transition energy, thus a beam which is unstable before crossing the transition becomes stable later. Also, if we had taken different values of ω_0 for the two particles (because of octupole effect, for instance) we would introduce some damping in the instability (Landau damping). The possibility of introducing this damping is not considered in this paper.

2. THE MAGNET LAMINATION

In this section we calculate the wake function ρ assuming that the coupling between the betatron oscillations of a particle and the surrounding media is predominantly caused by the lamination of the magnets.

We assimilate each crack of the lamination to a cavity. Calling ω_r its resonant frequency, Γ its decay time related to the loss factor Q by $\Gamma = \omega_r/2Q$, and Z_0 its shunt impedance, we can calculate first the voltage induced by a single particle and then, by means of the deflection theorem,[7] the force acting on another particle coming later at the time distance t. We have, in conclusion,

$$\rho(t) = \frac{r_o c^3}{d^2} \frac{\beta}{\gamma} \frac{\omega_r^2}{\omega_r^2+\Gamma^2} \frac{Z_o}{h} \alpha\, e^{-\Gamma t} \left(\sin \omega_r t - \frac{\cos \omega_r t}{4Q^2} \right) H(t) \qquad (7)$$

where β and γ are, respectively, the velocity and energy relativistic factor, $r_o = 1.53 \times 10^{-16}$ cm is the proton electromagnetic radius, 2d the magnet aperture, and

$$H(t) = 1 \quad \text{for} \quad t > 0$$
$$= 0 \quad \text{for} \quad t < 0.$$

Besides h is the distance between two consecutive cracks, and α is the fraction of the accelerator circumference occupied by the lamination. In other words, $Z_o\alpha/h$ is the average impedance per unit length.

From another paper[8] we know that the lamination in the booster has, actually, several resonant modes. If we take, as a good approximation, only the first of these modes we have from the same paper

$$\omega_r \sim 300 \text{ MHz}$$
$$Q \sim 1$$
$$Z_o\alpha/h \sim 5 \times 10^{-13} \text{ cm}^{-2} \text{ sec}$$

from which

$$\rho(t) = 184 \frac{\beta}{\gamma} e^{-\Gamma t} \left(\sin \omega_r t - \frac{\cos \omega_r t}{4} \right) H(t) \text{ sec}^{-2}$$

which in the case $\omega_r t \ll 1$ writes also

$$\rho(t) \approx -\rho_o H(t) \tag{8}$$

with

$$\rho_o = 46\ \beta/\gamma\ \ \text{sec}^{-2}.$$

If the bunch length is not too large, Eq. (8) is a good approximation of the wake form for our case.

The coefficient ρ_o is a function of time through the factor β/γ. Nevertheless, since we plan to neglect the variation with the energy of the beam size and length, in this approximation, we can replace ρ_o with its average over the acceleration cycle and take, for the booster case

$$\rho_o \sim 10\ \text{sec}^{-2}.$$

3. THE BEAM BEHAVIOUR VERSUS TIME

From M. Sands' paper we learn that in the model where all the particles have the same synchrotron oscillation amplitude A (in unit of time) the local coherent elongation of the beam bunch Z is a function of the time and of the phase ϕ, characteristic of the initial conditions of the synchrotron motion, namely

$$Z(\phi,t) = \sum_{n=0}^{\infty} a_n(t) e^{in\phi}$$

where

$$a_n(t) = a_n(0) e^{\,i\int_0^t \mu_n(t)\ dt} \tag{9}$$

with $\mu_n(t)$ = the collective frequency of the n^{th} mode of the transverse oscillations:

$$\mu_n(t) = \frac{N\rho_o}{8\pi^2\omega_o} F_n(x) \tag{10}$$

$$x = 2\omega_o A \frac{\xi}{\alpha} \tag{11}$$

N is the number of particles per bunch. The form of $F_n(x)$ depends on the coupling between the beam and the surrounding media. In our case

$$F_n(x) = \int_{-\pi}^{+\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ix\sin\left|\frac{\psi}{2}\right|\cos\phi} e^{in\psi} d\phi d\psi$$

$$= \mathcal{R}_n(x) - i\,\mathcal{J}_n(x) \tag{12}$$

with $\mathcal{R}_n$ and $\mathcal{J}_n$ real functions of x.

We can show that

$$\mathcal{R}_n(x) = 2\left[\pi\, J_n\left(\frac{x}{2}\right)\right]^2 \tag{13}$$

and

$$\mathcal{J}_n(x) = 4\pi(-1)^n \int_0^{\pi/2} \mathcal{H}_o(x\cos\theta)\cos 2n\theta d\theta \tag{14}$$

where J_n is the Bessel function of first kind and n^{th} order, and $\mathcal{H}_o$ is the Struve function of zeroth order.

We have the properties

$$\mathcal{R}_n(x) = \mathcal{R}_n(-x), \qquad \mathcal{J}_n(x) = -\mathcal{J}_n(-x)$$

and

$$F_n(x) = 0 \quad \text{for} \quad x = 0 \qquad \text{and } |x| \to \infty.$$

The following approximation can be used for small argument x.

$$\mathcal{J}_n(x) \approx \frac{8x}{1-4n^2} \quad . \tag{15}$$

Also, for practical purposes, $\mathcal{H}_o(x)$ can be approximated by

$$\mathcal{H}_o(x) \sim \frac{4}{\pi} J_1(x)$$

which gives

$$J_n(x) = 8x\, j_n\left(\frac{x}{2}\right)\, y_{n-1}\left(\frac{x}{2}\right)$$

$$= 8x\, G_n\left(\frac{x}{2}\right) \tag{16}$$

where j_n and y_n are the spherical Bessel functions of n^{th} order and, respectively, of the first and second kind. In particular, it is

$$G_0(z) = \left(\frac{\sin z}{z}\right)^2$$

$$G_1(z) = \frac{\cos z}{z}\left(\frac{\cos z}{z} - \frac{\sin z}{z^2}\right)$$

$$G_2(z) = \left(\frac{\cos z}{z^2} + \frac{\sin z}{z}\right)\left(\frac{3}{z^2}\cos z - \frac{3-z^2}{z^3}\sin z\right)$$

$G_0(z)$ is a positive defined function. $G_1(z)$ and $G_2(z)$ are negative for small z. In particular,

$$G_1(0) = -\frac{1}{3}, \qquad G_2(0) = -\frac{1}{15} .$$

Then, for large z, $G_1(z)$ and $G_2(z)$ switch signs and thereafter become positive defined. This is true for any G_n with $n \neq 0$, although the switching value of z increases with the mode number.

The sign of the function G_n states if the oscillation with the mode number n is stable or not, as we can see from Eqs. (9, 10, 12, and 16). For a fixed value of x, let us say positive, the first few modes (certainly n = 0) are unstable and the others above stable. It is true the vice-versa if x negative. In fact, the following property holds

$$G_n(x) = G_n(-x)$$

for whatever n.

x is a function of the time which becomes infinitely large as the transition energy and switches sign crossing the transition.

If we write

$$\alpha = \frac{1}{\beta^2}\left(\frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}\right)$$

where $\gamma_T = \gamma$ at the transition, and

$$\xi = \frac{1}{\beta^2}\ \frac{\Delta\nu/\nu}{\Delta p/p} = \frac{\xi_o}{\beta^2}$$

and we assume ξ_o constant, we have

$$x = \frac{2\omega_o A}{\frac{1}{\gamma_T{}^2} - \frac{1}{\gamma^2}}\ \xi_o\ . \tag{17}$$

If we assume the "missing bunches" observed in the booster before the transition energy are caused by the "head-tail" effect, we infer that actually ξ_o is a negative quantity.

In this case we can say that at the injection x is a small quantity and positive; only the mode n = 0 (corresponding to the center of mass oscillations) is unstable. Approaching the transition, x increases and other modes (in order n = 1, 2, ...) become unstable. At the transition, x is infinitely large and all the modes are unstable.* As soon as the transition is crossed, x is still very large but now negative, so that all the modes are stable. Proceeding further in the acceleration, x remains negative but its absolute value decreases so that the large-order modes become unstable, whereas the low-order modes (certainly n = 0) remain stable.

*Here we are neglecting that the motion around the transition energy is not adiabatic.

4. SOME NUMBERS

The instantaneous growth rate per particle is obtained from Eqs. (9, 10, 12, and 16)

$$p_n(t) = \frac{\rho_o}{\pi^2 \omega_o} \int_o^t xG_n \left(\frac{x}{2}\right) dt \tag{18}$$

where x is given by Eq. (17).

For the case of the booster we have

$\rho_o = 10 \text{ sec}^{-2}$ (see Section 2)

$\omega_o = 26.8 \times 10^6 \text{ sec}^{-1}$

$A = 10^{-9}$ sec

and

$$\gamma = \gamma_T - \hat{\gamma} \cos \frac{\pi t}{2t_T}$$

where

$\gamma_T = 5.446$

$t_T = 17.327$ msec

$\hat{\gamma} = 4.233$

which correspond to an acceleration up to 8 GeV and lasting 33.333 msec.

The growth rate per particle $p_n(t)$ is plotted in Figs. 1, 2, 3 and 4 for n = 0, 1, 2 and some values of ξ_o. We see that the fundamental mode n = 0 is the only one that can cause the beam to blow up.

In an experiment[2] it was found that a threshold current exists such that the "missing bunches" effect disappears below that current. The threshold value is 12 mA, corresponding to $N = 1.4 \times 10^9$. We believe that this threshold is not caused by any damping mechanism (Landau damping, for instance) since the very small ν-spread in the beam due to the octupole field. We think instead it is the effect of the transverse aperture of the machine. At the fundamental mode n = 0 the beam can blow up only up to a maximum size which depends on the beam intensity, and occurs at the transition. In Table 1 we have reported some numbers; t_T is the instant of the transition crossing, and $a_o(t_T)/a_o(0)$ is the ratio of the maximum beam size to the initial beam size. A beam pulse can be accelerated through the transition energy for a reasonable low intensity such that $a_o(t_T)/a_o(0)$ is less than $b/a_o(0)$, where b is the real aperture of the ring.

In the booster the ratio b/a (0) can be worth 2 for a good closed orbit in the vertical plane, and even less in the horizontal plane where b is essentially due to the separatrix of the rf buckets. These numbers seem to be in agreement with those in Table 1.

Table 1

			$a_o(t_T)/a_o(0)$	
ξ_o	$p_o(t_T)$	$Np_o(t_T)$	ρ_o=10 sec^{-2}	ρ_o=20 sec^{-2}
-0.1	0.10×10^{-9}	0.14	1.15	1.32
-0.3	0.19×10^{-9}	0.27	1.31	1.72
-0.6	0.27×10^{-9}	0.38	1.46	2.13
-1.0	0.35×10^{-9}	0.49	1.63	2.66

$N = 1.4 \times 10^9$, corresponding to a beam of 12 mA.

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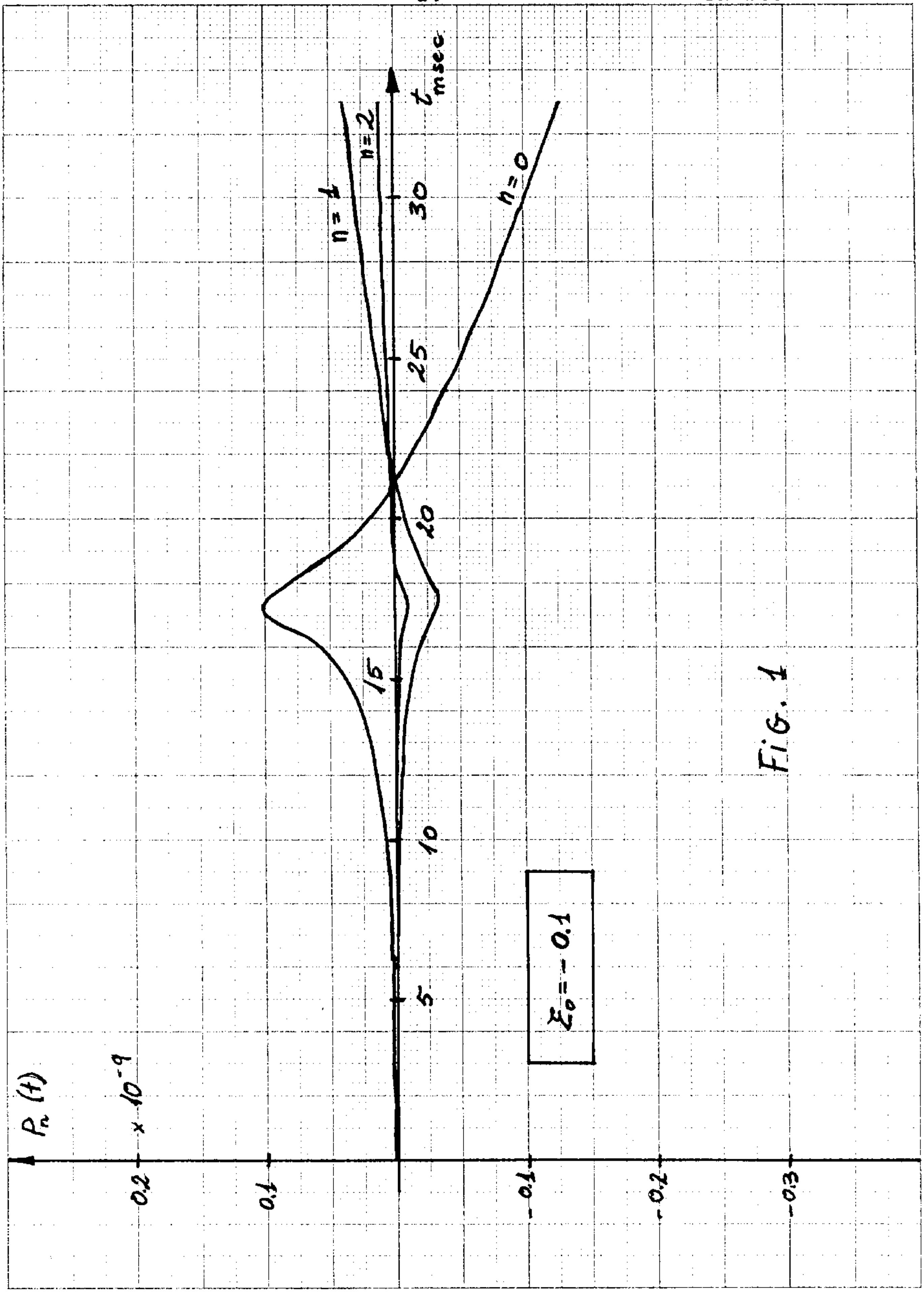

t_{msec}
$P_n(t)$
$\times 10^{-9}$
$n=1$
$n=2$
$n=0$
5
10
15
20
25
30
0.2
0.1
-0.1
-0.2
-0.3
$\mathcal{E}_0 = -0.1$

FIG. 1

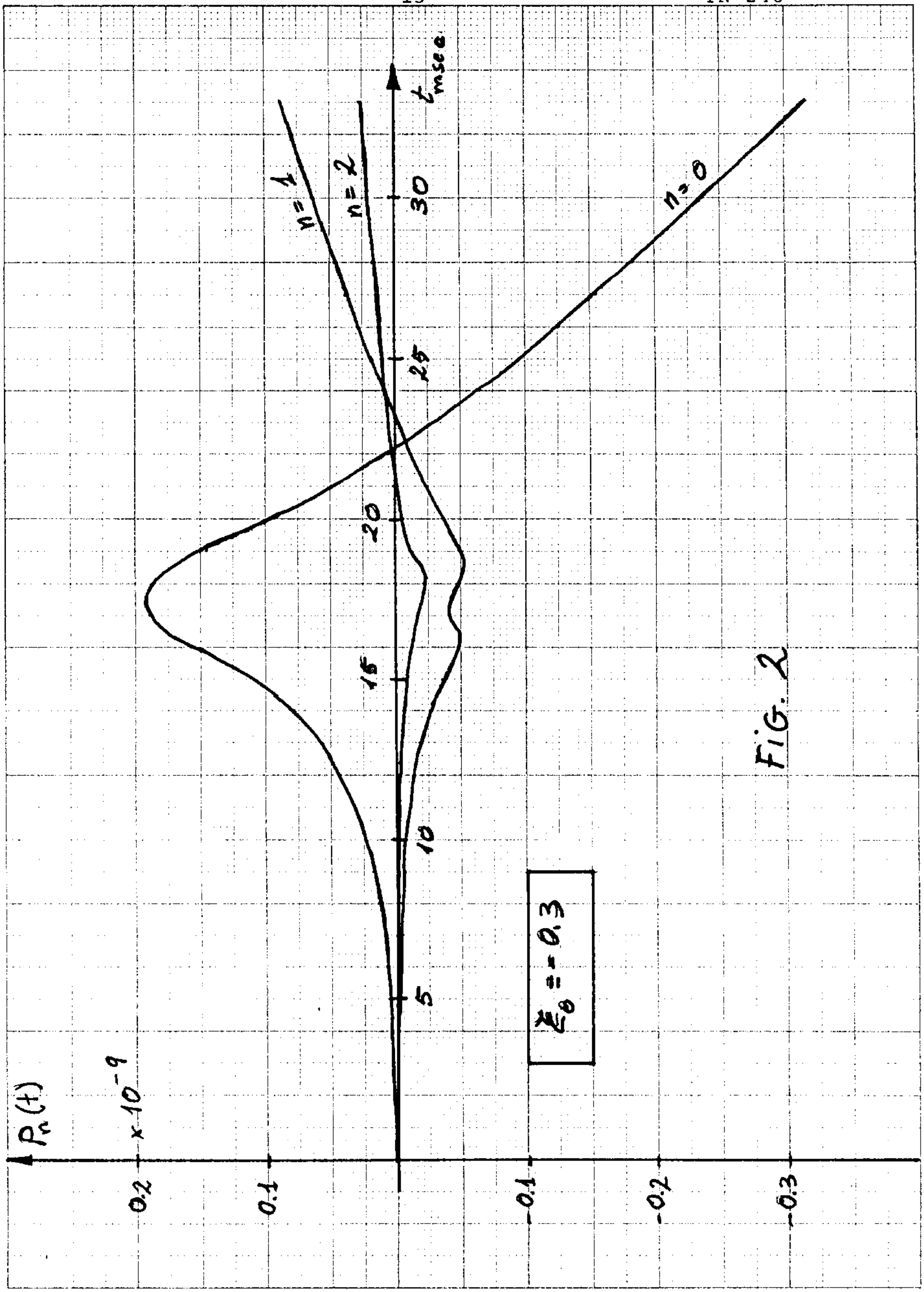

FIG. 2

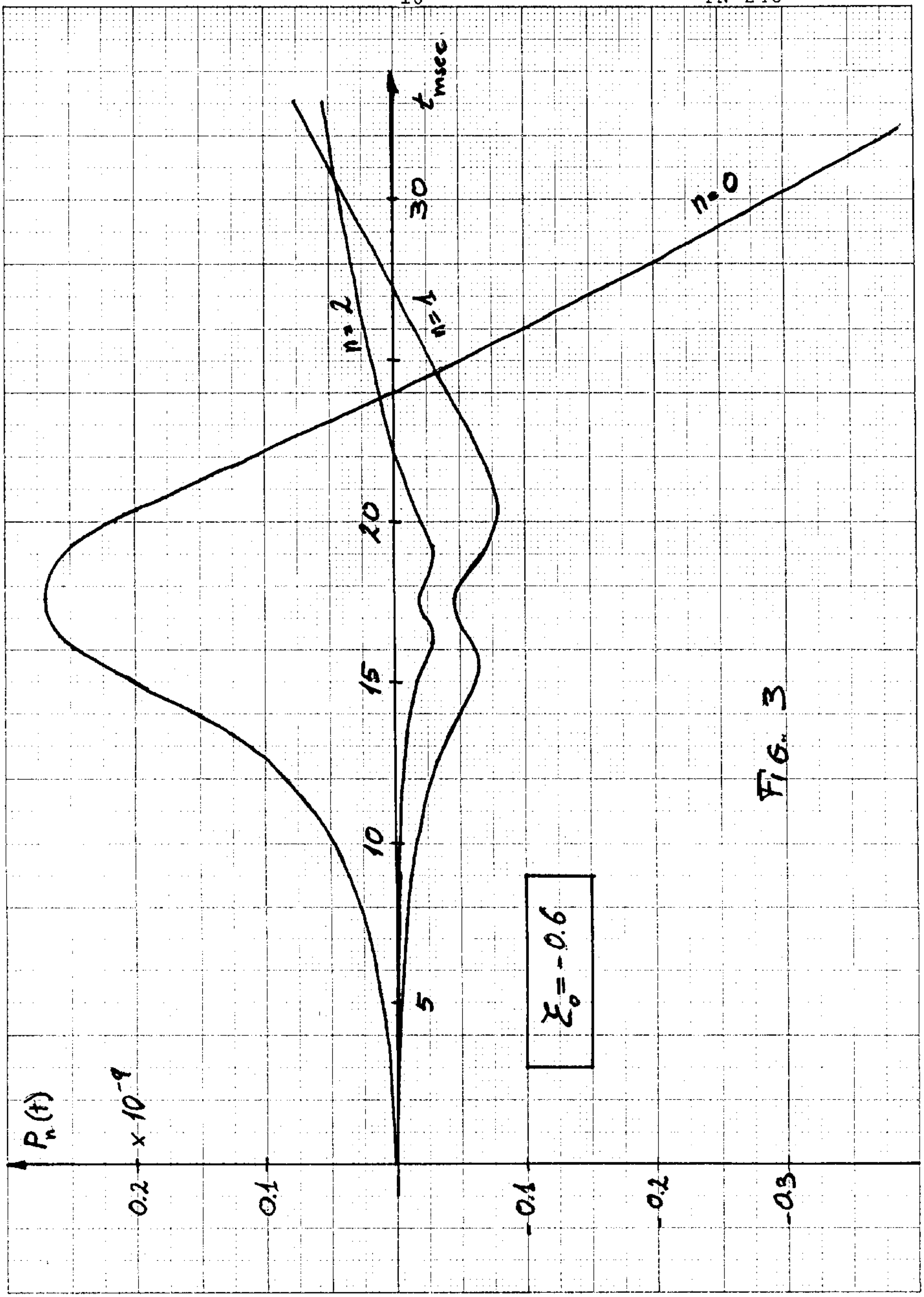

FIG. 3

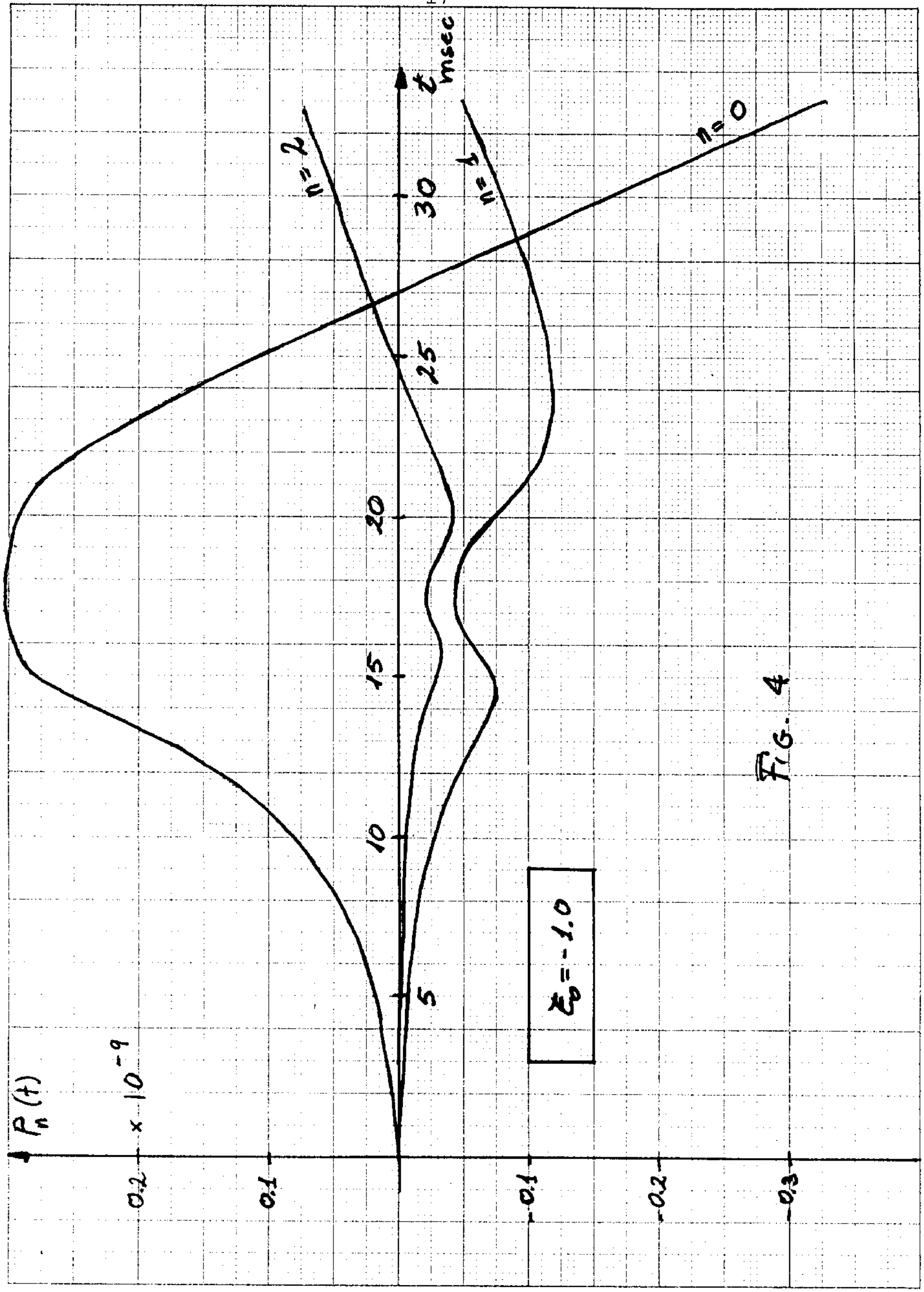
t msec
$n = 2$
$n = 1$
$n = 0$
30
25
20
15
10
5
$P_n(t) \times 10^{-9}$
0.2
0.1
-0.1
-0.2
-0.3
$\varepsilon_0 = -1.0$

FIG. 4